Estimation of Multicrop Production Functions

Richard E. Just, David Zilberman, and Eithan Hochman

This paper considers whether separability or nonjointness is the better approach for attaining tractability for multicrop production function estimation. Characteristics of agricultural production associated with allocated inputs, physical constraints, and output determination imply sufficient nonjointness for estimation, whereas separability is less plausible. The paper also addresses estimation of production functions with allocated inputs where allocations are not observed and demonstrates a proposed approach by way of example.

Key words: multioutput production, nonjointness, separability.

Perhaps the most difficult problem in estimating nonexperimental agricultural production functions is that input data typically are not available by crop. A farmer generally grows several crops, but the allocation of inputs among crops is not recorded. The most popular approach to this problem in recent econometric studies of agricultural production functions has been to use single-equation joint production functions which specify relationships between output quantities and aggregate input quantities or to use corresponding relationships between quantities and prices resulting from duality under (expected) profit maximization (Shumway and Chang, Weaver, Whittaker).[1] This paper investigates the applicability of such approaches in modeling agricultural production.

Shumway, Pope, and Nash recently addressed the problem of jointness in agriculture, concluding that various approaches are needed depending on the source of jointness. They suggest that allocable fixed inputs are an important source of jointness and show that the dual approach to production has serious limitations because it does not yield allocation equations (even if allocations are observed). Duality does not yield a complete solution of the production problem and, in particular, it does not provide information needed by decision makers who must make allocation decisions. Duality also does not yield a sufficient empirical framework for analysis of policies relating to inputs on specific crops, such as a wheat acreage policy, unless such policies are reflected in sample data.

This paper points out some corresponding limitations of the primal approach in its customary use. It then proposes a method to deal with these problems. The proposed approach is closely related to earlier work of Marschak and Andrews; Mundlak and Hoch; and Zellner, Kmenta, and Dreze. However, it also considers the missing data problem (the allocations) and how to attain more efficiency in estimation.

The methodology is based on the following assumptions that seem to characterize most agricultural production:

(*a*) *Allocated inputs.* Most agricultural inputs are allocated by farmers to specific production activities. For example, tractor and labor hours, fertilizer, and pesticides are allocated among wheat, corn, and soybean fields.

(*b*) *Physical constraints.* Physical constraints limit the total quantity of some inputs that a farmer can use in a given period of time. For example, land is often available in fixed amounts in given time periods.

(*c*) *Output determination.* Output combinations are determined uniquely by the allocation of inputs to various production activities

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[1] Evidence of this practice in the general economics literature can also be found in studies such as Powell and Gruen, and Mundlak.

aside from random, uncontrollable forces. For example, a farmer cannot change the output mix merely by adjusting some dials once all input allocations are determined. Alternatively, the mix of, say, wheat and corn produced on a farm is determined by the land, fertilizer, water, labor, tractor hours, etc., that are allocated to each enterprise.

Considering these structural features, some implicit assumptions of traditional approaches become clear. Then, a more comprehensive specification approach depending on data availability is proposed. Finally, a specific estimation technique is proposed to make complete use of available data and a priori specification when producers maximize profits and variable input allocations are unobserved. The method is demonstrated with data on Israeli desert agriculture. Results show how standard accounting data can be used to investigate variation in management efficiency and technology among farms and over time. The paper also demonstrates how unobserved input allocations can be estimated.

A Critique of Multiple-Output Production Functions

A multiple-output production function is a technical relationship that specifies possible output mixes that can be produced from each mix of inputs. Some examples include the generalization of the Cobb-Douglas function proposed by Klein:

(1) $$y_1 y_2^{\delta} = \alpha_0 x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3},$$

and the constant elasticity of transformation function:

(2) $$(\delta_1 y_1^{c} + \delta_2 y_2^{c})^{1/c} = g(x_1, x_2, x_3),$$

where outputs are denoted by y_k and inputs by x_j. A more popular model of multiple-output technology in agriculture is the constraint structure of a programming model,

(3) $$\mathbf{Ay} \le \mathbf{x},$$

where x and y are input and output vectors, respectively. In each case, distinct output trade-offs are available for a given set of aggregate inputs.

However, for allocable inputs, a fundamental difference exists between equation (3) and equations (1) and (2). The difference is that equation (3) shows specifically how the allocation of inputs among alternative production activities affects outputs, i.e.,

(4) $$y_k = \min_j \left\{ \frac{x_{kj}}{a_{jk}} \right\},$$

where $\mathbf{A} = \{a_{ij}\}$, $\mathbf{y} = (y_1, \ldots, y_k)'$, and x_{kj} is the amount of input j allocated to production of output k. If one knows how much of each input is allocated to each of the production activities (represented by columns of $\mathbf{A}$), then one can determine uniquely how much of each output is produced. Hence, the choice of output mix is completely determined by the allocation of inputs. A major reason for the popularity of (3) is that it is consistent with the three assumptions above.

To consider the consistency of (1) and (2) with these assumptions, suppose that farmers choose the amounts of each input applied to each output activity, e.g., the amount of fertilizer or tractor hours applied to corn land and the amount of fertilizer or tractor hours applied to wheat land. In this case, the inputs in (1) and (2) must be represented as

$$x_j = x_{1j} + x_{2j},$$

where x_{kj} is the quantity of the jth input applied in producing the kth output. Thus, for equation (1),

$$y_1 y_2^{\alpha} = \alpha_0 (x_{11} + x_{21})^{\alpha_1} (x_{12} + x_{22})^{\alpha_2} (x_{13} + x_{23})^{\alpha_3}.$$

The absurd implication of this relationship is that increasing the amount of fertilizer applied to wheat production offers the farmer a choice of, say, either increasing wheat production or corn production. Thus, the assumption of output determination cannot hold. The same is true for (2). But farmers must allocate physical inputs to distinct plots whether or not corresponding data are recorded. Because plots usually are used to grow only one crop at a time, these arguments imply that any relationship of the form

(5) $$f(y_1, \ldots, y_K) = g(x_1, \ldots, x_J)$$

lacks detail as a basis for either econometric or economic analysis of crop production. At a minimum, multiple-output production functions for agricultural crops must be consistent with aggregation of production activities and input allocations over different plots (crops).

The absurd implication of (1) and (2) for the agricultural case follows not from the use of a joint production possibilities frontier [(3) also yields a nontrivial frontier] but from the restriction that this frontier is separable with respect to inputs and outputs. To see this, consider a general multi-output production function given by $h(\mathbf{y}, \mathbf{X}) = 0$ where h is a vector function, $\mathbf{y}$ is a $K \times 1$ vector of outputs, and $\mathbf{X}$ is $K \times J$ matrix with elements x_{kj} representing the allocation of inputs $\mathbf{x} = (x_1, \ldots, x_J)'$,

$$(6) \quad \sum_{k=1}^{K} x_{kj} = x_j, \qquad j = 1, \ldots, J.$$

Since this general form is not tractable for most purposes, the common approach has been to assume either separability of $h(\mathbf{y}, \mathbf{X})$ with respect to inputs and outputs, which implies that $h(\mathbf{y}, \mathbf{X}) = f(\mathbf{y}) - g(\mathbf{X})$ as in (5), or to assume some kind of nonjointness of the production technology (Hall, Lau). Given that some simplifying assumptions must be imposed, the argument here is that allocated inputs, physical constraints, and output determination better characterize agricultural production than separability.

Although the relationship in (3) is reasonable by this standard, its fixed proportions limitations impose nonjointness with respect to both inputs and outputs (Lau). The approach here is to generalize the technology matrix $\mathbf{A}$ column by column to allow input substitution yet retain additive physical accounting relationships for allocable inputs. Suppose h is twice differentiable and uniquely determines outputs $\mathbf{y}$ from inputs $\mathbf{X}$. Then, using this implicit function theorem, one can solve $h(\mathbf{y}, \mathbf{X}) = 0$ for a vector function $\mathbf{y} = f(\mathbf{X})$ assuming the relevant Jacobian exists. Hence, if inputs are allocated among production activities (input x_{kj} affects production of y_k but not y_{k^*}), one can write

$$(7) \quad y_k = f_k(x_{k1}, \ldots, x_{kJ}), \qquad k = 1, \ldots, K$$

so that production activities are linked only by the physical accounting relationships in (6).[2] Thus, under the assumptions of allocated inputs, physical constraints, and output determination, one obtains a production system where nonjointness is imposed only with respect to inputs and not outputs (Lau).

Under some circumstances, the production system in (6) and (7) can be represented by a relationship between outputs and aggregate inputs in the spirit of (1) and (2). For example, consider two outputs and one input with

$$(8) \quad \begin{aligned} y_1 &= f_1(x_{11}) \equiv A_1 x_{11}^{\alpha_1}, \text{ and} \\ y_2 &= f_2(x_{21}) \equiv A_2 x_{21}^{\alpha_2}. \end{aligned}$$

Then the choices for output mix with aggregate input x_1 is represented by

$$(9) \quad (A_1^{-1} y_1)^{1/\alpha_1} + (A_2^{-1} y_2)^{1/\alpha_2} - x_1 = 0.$$

However, where input allocations are observed, econometric analysis of (8) generally leads to much better estimates of production function parameters than (9).

Moreover, when (8) is generalized, such as with two inputs in each function, then one generally cannot obtain equations that involve only outputs and aggregate inputs and reflect only production technology as in (5) and (9), i.e., that do not include additional assumptions such as behavioral rules. To see this, note that (6) and (7) contain $J + K$ equations in $J \cdot K + J + K$ variables. According to the implicit function theorem, they can be solved in principle for any $J + K$ variables in terms of the other $J \cdot K$ variables (assuming the relevant Jacobians exist).[3] But only in very restrictive ways can this lead to a relationship not involving the allocations of inputs to individual production activities.[4] For instance, one of the resulting equations could be of the form

$$(10) \quad y_k = f_k(x_1, \ldots, x_J, y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_K)$$

only if the other $J + K - 1$ variables excluded from the right-hand side were the $J \cdot K$ variables representing input allocations (the x_{kj}). This would be possible only if $K = 1$ or $J = 1$ or if at least $J \cdot K - J - K + 1$ of the allocation variables in X do not appear in (6) and (7).[5] In other words, for general agricultural applications, equations such as (5) restrict drastically the flexibility offered by allocation decisions.

[2] If sufficient temporal and spatial detail is recorded in input data, then many inputs may apply only to a single production activity. Thus, the number of physical accounting relationships needed for empirical purposes may depend on the detail of recorded data.

[3] Note that existence of relevant Jacobians is assumed throughout the rest of this paper. Uniqueness follows under the usual assumptions of strong concavity and monotonicity on the f_k in (7).

[4] A similar point has been made by Mittelhammer, Matulich, and Bushaw, but their analysis stops short of providing a practical alternative approach. Their argument is that the production function must be represented by multiple equations as in the general vector function, $h(\mathbf{y}, \mathbf{X}) = 0$.

[5] Here we assume that inputs are homogenous so that linear aggregation is the only consistent method of aggregation (Green).

In fact, use of (5) or (10) is equivalent to imposing $J \cdot K - J - K + 1$ arbitrary restrictions in addition to the technological and physical relationships in (6) and (7).

If equations such as (5) and (10) are used, then the form and source of implicit additional restrictions should be made clear. One source of additional restrictions can be the behavioral rule followed by the decision maker—for example, the $J \cdot K$ first-order conditions of profit maximization. In this case, the relationship in (5) or (10) no longer can be regarded as purely technical. Rather, it will be a reduced form combining technical and behavioral relationships. Note also that first-order conditions contain additional variables (such as prices) which also must be eliminated in reaching the right-hand side of (10).

If the first-order conditions include $J \cdot K$ allocating equations with the K output prices in $p = (p_1, \ldots, p_K)'$ and J input prices in $w = (w_1, \ldots, w_J)'$, then (6) and (7) plus the first-order conditions constitute $K + J + K \cdot J$ equations in $2(K + J) + K \cdot J$ variables. Since profits are homogenous in prices, one of the price variables must be chosen as a *numèraire* to solve the equation system for any $K + J + K \cdot J$ variables. The number of restrictions is just sufficient to find a relationship such as (5) or (10).

An alternative approach to obtaining the additional restrictions without introducing price variables is to use the $J \cdot K$ first-order conditions for profit maxmization to solve for $(K - 1) \cdot (J - 1)$ profit maximization conditions not involving prices,

$$(11) \qquad MRT_{y_1 y_{k^*}; x_1} = MRT_{y_1 y_{k^*}; x_j}, \quad k^* = 2, \ldots, K; \quad j = 2, \ldots, J$$

or

$$(12) \qquad MRTS_{x_1 x_{j^*}; y_1} = MRTS_{x_1 x_{j^*}; y_k}, \quad k = 2, \ldots, K; \quad j^* = 2, \ldots, J$$

where

$$MRT_{y_1 y_k; x_j} = \frac{\partial y_1 / \partial x_j}{\partial y_k / \partial x_j},$$

$$MRTS_{x_1 x_j; y_k} = \frac{\partial y_k / \partial x_1}{\partial y_k / \partial x_j}.$$

Combining these conditions with (6) and (7) gives $(K - 1) \cdot (J - 1) + K + J$ equations in $K + J + K \cdot J$ variables. Without redundancy, this approach is also just sufficient to obtain (5) or (10) by the implicit function theorem.

While first-order conditions can lead to general forms such as (5) and (10), one also must recall that arbitrary specifications of (5) can be inconsistent with first-order conditions for problems with allocation decisions. One must investigate the source and implication of specifications like (5) for problems with allocation decisions. For example, two sets of implicit restrictions which can be associated with (1) and (2) are given in (11) and (12). That is, no input or output choices in (1) or (2) can cause failure of (11) and (12). Thus, (1) and (2) may be applicable for some allocation problems under profit maximization but inappropriate under other behavioral rules. The assumption of profit maximization may not be unreasonable, but at least it should be recognized as implicit in (1) and (2) in this context.

A more serious problem is that some restrictive assumptions about technology must be imposed to obtain functions such as (1), (2), or (5). To see this for a production problem with allocable inputs, suppose that the underlying technology in (7) is given by

$$y_k = x_{k1}^{\beta_{k1}} x_{k2}^{\beta_{k2}} x_{k3}^{\beta_{k3}}, \quad k = 1, 2.$$

Then

$$(13) \qquad \frac{\partial y_k}{\partial x_{kj}} = \beta_{kj} \frac{y_k}{x_{kj}}, \quad k = 1, 2; \quad j = 1, 2, 3$$

which, together with (6), is consistent with (1) for $\beta_{1j} = \alpha_j$ and $\beta_{2j} = \alpha_j/\delta, j = 1, 2, 3$. That is, all production elasticities for one product are proportional to those for the other product. By substituting (13) into (11) or (12), one can verify that $\beta_{1j} = \delta\beta_{2j}, j = 1, 2, 3$, for some δ, and these are exactly the conditions under which (11) and/or (12) hold. But these restrictions would exclude the possibility that the relative productivity of land in cotton versus corn may be higher than for fertilizer because of a cotton acreage restriction. Similar arguments can be establshed for (2). These results reveal that some common multi-output production functions have serious limitations for agricultural applications.

Implications for Specification and Estimation

To illustrate the implications of these results for specification and estimation, consider the five general data groups represented by **y**, **x**, **X**, **p**, and **w**. The data missing in most cases are allocations in **X**. However, the quality and availability of input price data (**w**) or aggregate input quantity data (**x**) also may be limited. Thus, this section considers several cases of data availability. For each case, equations (6) and (7) are assumed to underlie the allocative problem.

First, one can estimate the production relations in (7) directly if and only if data are available on both input allocations and output (**X** and **y**). This is the only general input allocation case where technological relationships can be estimated without implicit assumptions that restrict either behavior or technology. Avoiding assumptions about behavior in production function estimation is desirable if one wishes to investigate behavioral criteria specifically (Just and Pope). However, increased efficiency in estimating production parameters can be attained with additional nonredundant relationships reflecting plausible behavioral rules.

To consider the case where the allocations in **X** are not available (or where the efficiency associated with behavioral specification is desirable), suppose the production problem is one of profit maximization,

$$\max_{\mathbf{y},\mathbf{X},\mathbf{x}_f} \; \mathbf{p}'\mathbf{y} - \mathbf{w}'\mathbf{x}_v$$

subject to

$$\mathbf{y} = f(\mathbf{X}),$$

$$\mathbf{X}\mathbf{e} = \mathbf{x},$$

where $\mathbf{x}' = (\mathbf{x}'_f, \mathbf{x}'_v)$ so that $\mathbf{x}_f$ is the $J_f \times 1$ subvector of aggregate input uses corresponding to fixed or constrained inputs; $\mathbf{x}_v$ is a similar $J_v \times 1$ subvector corresponding to unconstrained variable inputs, and $\mathbf{e} = (1, 1, \ldots, 1)'$. The associated Lagrangian,

$$L = \mathbf{p}'\mathbf{y} - \mathbf{w}'\mathbf{x}_v - \boldsymbol{\lambda}'[\mathbf{y} - f(\mathbf{X})] - \boldsymbol{\phi}'(\mathbf{X}\mathbf{e} - \mathbf{x}),$$

has first-order conditions,

$$(14) \qquad \frac{\partial L}{\partial \mathbf{y}} = \mathbf{p} - \boldsymbol{\lambda} = \mathbf{0} \qquad [K]$$

$$(15) \qquad \frac{\partial L}{\partial \mathbf{x}_v} = \boldsymbol{\phi}_v - \mathbf{w} = \mathbf{0} \qquad [J_v]$$

$$(16) \qquad \frac{\partial L}{\partial \mathbf{X}_{\cdot j}} = \boldsymbol{\lambda}' \frac{\partial f}{\partial \mathbf{X}_{\cdot j}} - \phi_j \mathbf{e} = \mathbf{0}, \quad j = 1, \ldots, J \; [K \cdot J]$$

$$(17) \qquad \frac{\partial L}{\partial \boldsymbol{\lambda}} = f(\mathbf{X}) - \mathbf{y} = \mathbf{0} \qquad [K]$$

$$(18) \qquad \frac{\partial L}{\partial \boldsymbol{\phi}} = \mathbf{x} - \mathbf{X}\mathbf{e} = \mathbf{0} \qquad [J]$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\phi} = (\boldsymbol{\phi}'_f, \boldsymbol{\phi}'_v)' = (\phi_1, \ldots, \phi_J)'$ are vectors of shadow prices for outputs and inputs (with the partitioning of $\boldsymbol{\phi}$ corresponding to the partitioning of **x**), and $\mathbf{X}_{\cdot j} = (x_{1j}, \ldots, x_{Kj})'$. The conditions in (14)–(18) give $2K + J + J_v + K \cdot J$ nonredundant equations (the number of equations in each relationship is given in brackets) in $3K + 2J + J_v + K \cdot J - 1$ variables ($\boldsymbol{\lambda}$, $\boldsymbol{\phi}$, **X**, $\mathbf{x}_v$, **y**, **p**, **w**, and $\mathbf{x}_f$). Note that zero degree homogeneity in prices implies that (**p**, **w**) contains only $J_v + K - 1$ exogenous variables. In general, no other nonredundant equations will add more information for parameter estimation.

The number of observable equations that can be determined from (14)–(18) depends on how many variables are observed. According to the implicit function theorem, the number of nonredundant equations that can be expressed solely with observable variables is, at most, the number of observable variables less $J + K - 1$ (the number of exogenous variables). Thus, if one can find at least this many nonredundant equations that include no unobservable data, then efficient (full information) estimation is possible.

To consider a specific case of data availability, suppose first that all data other than shadow prices are observed. Then $K + J$ equations determining $\boldsymbol{\lambda}$ and $\boldsymbol{\phi}$ can be discarded. The maximum number of nonredundant equations for this case is attained by the system

$$\mathbf{p}' \frac{\partial f}{\partial X_{\cdot 1}} - \mathbf{p}' \frac{\partial f}{\partial \mathbf{X}_{\cdot j}} = \mathbf{0}, \quad j = 2, \ldots, J_f \; [(J_f - 1) \cdot K]$$

$$(19) \qquad \mathbf{p}' \frac{\partial f}{\partial \mathbf{X}_{\cdot j}} - w_j \mathbf{e} = 0, \quad j = J_{f+1}, \ldots, J \; [J - K]$$

$$f(\mathbf{X}) - \mathbf{y} = \mathbf{0} \qquad [K]$$

$$\mathbf{x} - \mathbf{X}\mathbf{e} = \mathbf{0}. \qquad [J]$$

With addition of stochastic disturbances, all of these equations can be used in estimation of the parameters of a multioutput production problem with allocations.[6] Collectively, they

[6] Of course, if some equations do not have stochastic distur-

exhaust the a priori theoretical information.

Second, suppose neither shadow prices nor allocations of inputs ($\boldsymbol{\lambda}$, $\boldsymbol{\phi}$, $\mathbf{X}$) are observed. Then, the maximum number of observable equations is $K + J_v$ (the number of observable variables in **y**, **x**, **w**, and **p** after correcting for homogeneity less the $K + J - 1$ exogenous variables). One set of equations which reflects full information follows from restricted duality,

$$\text{(20)} \qquad \begin{aligned} \mathbf{y} &= \mathbf{y}(\mathbf{p}, \mathbf{w}, \mathbf{x}_f) && [K] \\ \mathbf{x}_v &= \mathbf{x}_v(\mathbf{p}, \mathbf{w}, \mathbf{x}_f). && [J_v] \end{aligned}$$

Another set of equations reflecting the same full information is represented by

$$\text{(21)} \qquad \begin{aligned} y_k &= y_k(y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_K, \mathbf{x}) && [1] \\ \mathbf{p} &= \mathbf{p}(\mathbf{y}, \mathbf{x}) && [K] \\ \mathbf{w}^* &= \mathbf{w}^*(\mathbf{y}, \mathbf{x}) && [J_v - 1] \end{aligned}$$

where $\mathbf{w}^*$ is the non-*numèraire* component of **w**, and some w_j is a *numèraire*. Note that the first equation of the system in (21) is a relationship, in the form of (10), with behavioral content. The remaining relationships are inverse supply and demand equations.

Third, if only **y** and **x** are observed, then full a priori theoretical information is contained in only one relationship such as (21). But in agriculture, input and output prices are usually observed. Because the associated equations in (21) share common parameters with the first equation, combined estimation is superior.

A fourth reasonable case would be where allocations, totals of variable inputs, allocations of fixed or constrained inputs, and shadow prices are not observed. According to the implicit function theorem, the number of observable equations is then K (the number of variables in **y**, **p**, $\mathbf{x}_v$, and **w** correcting for homogeneity minus $K + J - 1$). One possible complete system for this case is the first K equations in (20).

Many other data availability cases can be considered. This approach, however, gives a systematic way of knowing how many equations must be considered to reflect full a priori theoretical information in each case. Depending on data availability, the observable relationships may be insufficient to estimate all the parameters of the problem or any particular production function. Shumway, Pope, and Nash point out that estimates of the standard duality equations in (20) are not sufficient to estimate input allocations or output specific input parameters. However, the answers to questions of this kind for other relationships depend on functional forms. They must be investigated case by case.

Note that all of these arguments generalize easily for production under uncertainty. Under expected profit maximization with random prices and production where, for example, $\mathbf{y} = \boldsymbol{\epsilon} \cdot \bar{\mathbf{y}}$, $E(\boldsymbol{\epsilon}) = \mathbf{I}$, $\bar{\mathbf{p}} = E(\boldsymbol{\epsilon}'\mathbf{p})$, and $E(\mathbf{w}) = \bar{\mathbf{w}}$, all of the relationships in (19)–(21) follow if random disturbances associated with **y**, **p**, and **w** are inserted appropriately.[7]

Detailed Specification for a Common Case

Perhaps the most common case of data availability in agriculture is where total use of variable inputs, such as water and fertilizer, is observed but their allocations to various crops are not. On the other hand, allocations of the major fixed factor, land, are observed. Input and output prices and production are generally observable. Thus, a full information estimation approach must utilize the observed land allocations and compensate for the lack of information on allocations of other inputs. This section proposes such an approach. The approach is demonstrated explicitly with Cobb-Douglas specifications for the production functions in (7). Note that this specification thus assumes neither separability of inputs and outputs nor output nonjointness. The stochastic nature of agricultural production is incorporated explicitly by allowing random production and output prices in the expected profit maximization context.

Suppose the production functions in (7) are of the form

$$y_{ikt} = \prod_{j=1}^{J} x_{ijkt}^{\alpha_{jk}} e^{\beta_{kt} + \gamma_k m_i + \epsilon_{ikt}^y},$$

where t represents time; i denotes farmer; m_i is a human capital measure for farmer i; and $\epsilon_{ikt}^y \sim N(0, \sigma_k)$.[8] The α_{jk} are production elas-

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bance, then they may be treated as identity equations and not be incorporated in the ultimate estimation procedure. But such an assumption may not be reasonable for some of the equations. For example, a stochastic disturbance in $f(\mathbf{X})$ may cause other first-order conditions involving derivatives of $f(\mathbf{X})$ to be stochastic.

[7] Note that $\boldsymbol{\epsilon}$ is a disturbance matrix which may be diagonal. The kth diagonal element is simply a multiplicative disturbance for the kth production function.

[8] The m_i variable can also be called management where management is regarded as a technical input following Mundlak.

ticities for input j in crop k, β_{kt} is a technology/weather effect for crop k at time t, and γ_k is the effect of human capital in production of crop k. Suppose output price is random with $p_{kt} = p_k(Z_{it}) \cdot e^{\epsilon_{ikt}^p}$, where $p_k(Z_{it})$ is expected output price at decision-making time based on information set Z_{it} and ϵ_{ikt}^p is jointly normally distributed with ϵ_{ikt}^y and has zero mean. Thus,

$$(22) \quad \bar{r}_{ikt} \equiv E(p_{kt}\, y_{ikt}) = p_k(Z_{it}) \cdot \prod_{j=1}^{J} x_{ijkt}^{\alpha_{jk}} e^{\beta_{kt}+\gamma_k m_i+\sigma_k^r/2},$$

where

$$\sigma_k^r = \text{var}(\epsilon_{ikt}^r),\ \epsilon_{ikt}^r = \epsilon_{ikt}^p + \epsilon_{ikt}^y.$$

Following the previous section, the first-order conditions corresponding to (14)–(18) for the case of expected profit maximization give $2K + J + J_v + K \cdot J$ nonredundant equations in $3K + 2J + J_v + K \cdot J - 1$ variables of which $J_v \cdot K + K + J$ variables ($\mathbf{X}_v$, $\boldsymbol{\lambda}$, and $\boldsymbol{\phi}$) are unobserved. After eliminating equations which determine unobserved variables (using the implicit function theorem), the number of nonredundant observable equations is $K + K \cdot J_f + J_v$. One set of relationships attaining this number of nonredundant equations is

$$(23) \quad x_{ijt} = \sum_{k=1}^{K} \alpha_{jk} \frac{\bar{r}_{ikt}}{w_{jt}} + \epsilon_{ijt}^x, \quad j = J_{f+1}, \ldots, J \quad [J_v]$$

$$(24) \quad \ln x_{ijkt} = \ln \alpha_{jk} - \ln \alpha_{j\kappa} + \ln\left(\frac{\bar{r}_{ikt}\, x_{ij\kappa t}}{\bar{r}_{i\kappa t}}\right) + \epsilon_{ijkt}^x$$

$$k = 1, \ldots, K;\quad k \neq \kappa;\quad j = 1, \ldots, J_f \quad [J_f \cdot (K-1)]$$

$$(25) \quad \ln y_{ikt} = \sum_{j=1}^{J_f} \alpha_{jk} \ln x_{ijkt} + \sum_{j=J_f+1}^{J} \alpha_{jk} \ln\left(\frac{\bar{r}_{ikt}}{w_{jt}}\right) + \tilde{\beta}_{kt} + \gamma_k m_i + \epsilon_{ikt}^y, \quad k = 1, \ldots, K \quad [K]$$

$$(26) \quad x_{ijt} = \sum_{k=1}^{K} x_{ijkt}, \quad j = 1, \ldots, J_f \quad [J_f]$$

where

$$\tilde{\beta}_{kt} = \beta_{kt} + \sum_{j=J_f+1}^{J} \alpha_{jk} \ln \alpha_{jk}.$$

Since variable input allocations are unobserved, equation (23) provides an aggregate input equation by summing the usual first-order conditions solved for allocations. Equation (24) is obtained by taking ratios of usual first-order conditions, solved for allocations, since shadow prices of fixed or constrained inputs are unobserved. Equations (25) are the usual log-linear production functions with unobservables replaced through first-order conditions.

Equations (23)–(26) give a system which is almost linear for estimation, given $\bar{r}_{ikt}$. Of course, actual data likely will not fit equations (23) and (24) exactly, so random errors in profit maximization are considered as in Marschak and Andrews; Mundlak and Hoch; or Zellner, Kmenta, and Dreze by inserting additive random disturbances (ϵ_{ijt}^x and ϵ_{ijkt}^x) in (23) and (24), $E(\epsilon_{ijt}^x) = E(\epsilon_{ijkt}^x) = 0$. Data typically fit equation (26) exactly, so it can be regarded as an identity.

The most difficult problem in joint estimation of equations (23)–(26) is the unobservability of $\bar{r}_{ikt}$. One possibility is to replace $\bar{r}_{ikt}$ using the equation $\bar{r}_{ikt} = r_{ikt} + r_{ikt}\, \delta_{ikt}$, where $\delta_{ikt} = e^{\sigma_k^r/2 - \epsilon_{ik}^{rt}} - 1$, $E(\delta_{ikt}) = 0$. Thus, equations (23)–(26) can be rewritten replacing $\bar{\mathbf{r}}$ by $\mathbf{r}$ and $\boldsymbol{\epsilon}$ by $\tilde{\boldsymbol{\epsilon}}$ where

$$(27) \quad \tilde{\epsilon}_{ijt}^x = \epsilon_{ijt}^x + \sum_{k=1}^{K} \alpha_{jk} \frac{r_{ikt}}{w_{jt}} \delta_{ikt},$$

$$(28) \quad \tilde{\epsilon}_{ijkt}^x = \epsilon_{ijkt}^x + \epsilon_{i\kappa t}^r - \epsilon_{ikt}^r + \left(\frac{\sigma_k^r - \sigma_\kappa^r}{2}\right),$$

$$(29) \quad \tilde{\epsilon}_{ikt}^y = \epsilon_{ikt}^y - \sum_{j=Jf+1}^{J} \alpha_{jk} \left(\epsilon_{ikt}^r - \frac{\sigma_k^r}{2}\right),$$

since

$$\ln \bar{r}_{ikt} = \ln r_{ikt} - \epsilon_{ikt}^r + \frac{\sigma_\kappa^r}{2}.$$

This transformation introduces several complications. First, the disturbances defined by (28) and (29) have nonzero expectations. In equation (29), the expectation of $\tilde{\epsilon}_{ikt}^y$ can be included in $\tilde{\beta}_{kt}$ (or a constant term) for estimation. In the case of equation (28), however, a constant term must be added in equation (24), after replacing $\bar{\mathbf{r}}$ by $\mathbf{r}$ to account for the nonzero expectation. These expectations correct for seemingly risk-averse behavior under profit maximization discussed by Just (1975).

A second complication introduced by (27) is heteroscedasticity in the aggregate variable input equations. This heteroscedasticity is not serious if (*a*) the production elasticities, the ratio of revenue to the input price, and the degree of revenue variability are similar across crops; (*b*) revenue is highly correlated among crops, and variability is similar across crops; or (*c*) the random disturbances in revenue are swamped by errors in choosing maximizing levels of variable inputs. A test of this hypothesis could not be rejected for the empirical case here.

A third complication is that endogenous variables appear on the right-hand side of (23)–(26) after replacing $\bar{\mathbf{r}}$ by $\mathbf{r}$. Hence, a simultaneous equation estimation procedure must be used. One possibility is to ignore the information contributed by the equations with nonlinearities in (24) and use linear simultaneous equations methods to estimate the remaining equations. However, this produces inefficient estimators. Another approach is to use nonlinear two-stage or three-stage least squares (N2SLS or N3SLS) on the entire system. The latter approach attains consistency and asymptotic efficiency regardless of correlations among disturbances in first-order conditions assuming homoscedasticity and no correlation across time and decision makers (Hausman). Note that relatively little nonlinearity is included in (23)–(26) so that the usual convergence problems tend not to occur. Furthermore, good starting values can be generated by linear 2SLS estimation of equations (23) and (25).

Application to Israeli Agriculture

The methodology of the previous section was applied to estimating production functions for individual farmers in southern Israel. The data were collected in a Moshav (cooperative village) in the Arava region of Israel. The Arava includes the plains between the Red Sea and the Dead Sea. It is an arid region with high minimum and maximum temperatures making it an off-season producer of winter vegetables for local and export markets. The Moshav consists of seventy small family farms (10 acres each) with cooperation, mutual collaboration, and the principle of self-employment as its ideological basis. Each farm is privately controlled, and its economic life depends on its profitability.

The data were collected from the Moshav's accounting system supported by data on land allocation from the extension service. The allocations of variable inputs to various crops were not recorded. Rather, the data included total expenditures or quantities of purchased inputs. Total revenues and quantities of outputs and land allocations were also observed. Specifically, the data were a combined cross-section and time-series yielding annual observations for 1977–80 including the following:

(*a*) Cultivated area, production, and revenue by crop for tomatoes, bell peppers, onions, melons, and eggplants. Almost all farmers grow tomatoes, bell peppers, and melons. A smaller number also grow eggplants and onions.

(*b*) Aggregate water input and expenditure. The arid conditions and remoteness of outside water make it one of the most critical inputs. Drip irrigation, a very efficient way to use water, is the dominant technology.

(*c*) Other purchased input expenses. These include variable costs of inputs such as fertilizer, pesticides, and cultivating materials measured in 1980 prices. For estimation, these data were used to construct an index of other variable input use.

(*d*) Management ability. To reflect differences in human capital, a Delphi panel was assembled from farm advisors and settlement leaders and used to assess management ability of individual farmers. The variable is a rating from one to ten representing consensus of the panel.

Other major inputs, such as labor and capital, were determined by the institutional setting. That is, when settlements are established by the Jewish Agency, all farmers receive the same machinery items, the same amount of land, etc. Furthermore, the Moshav constrains the amount of labor that can be hired so that neither labor nor capital differed substantially over either time or farms. These conditions both simplified and complicated the estimation process. Since aggregate labor and capital use were constant within the data, no equations were needed to explain aggregate labor and capital use. However, since allocations of labor and capital also were not observed, it was not possible to estimate the associated equations for the allocation of labor and capital to the various crops. It was also not possible to estimate the contribution of labor and capital to individual crops in the production functions.

One possible approach in this case is to reduce further the production system in equations (26)–(29) to obtain an estimable system of equations. However, some alternative assumptions seemed appropriate and preserve simplicity of interpretation of the estimated model.[9] That is, the tractor and equipment seemed to have excess capacity because of the small acreage held by each farm. The only other major capital input, drip irrigation equipment, is allocated to each crop in amounts proportional to the land used. Thus, the effect of capital was assumed to be reflected in the land coefficient. Labor input is less clear, but labor prior to harvest also appeared to be proportional to land. Harvest labor, on the other hand, had little effect on production. Family labor, supplemented by volunteer labor, is simply adjusted so that none of a crop is left unharvested. Thus, any labor effects in the production function (nonharvest labor) were assumed to be reflected in the land coefficient.

Using these data, equations (23)–(26) were estimated by nonlinear two-stage and three-stage least squares assuming constant returns to scale. No convergence problems were experienced presumably due to the model's near-linearity. The two-stage estimator converged in thirty iterations, and the three-stage estimator required six iterations. The model was estimated using the statistical analysis system (SAS) for less than $8 for both estimates combined. The results are reported in table 1. The results of the two estimation methods were quite similar. All coefficients have theoretically appropriate signs, and most have quite reasonable magnitudes. Except for tomatoes and melons, all of the water production elasticities were between .05 and .08; all of the other purchased input elasticities were between .22 and .44; and all implied nonpurchased (land-labor-capital) production elasticities were between .51 and .70. The tomato and melon production functions imply much less importance of purchased inputs. However, tomatoes are considered to be relatively more labor intensive, and melons to be a residual crop that is relatively more land intensive.

Both estimates show that human capital has a significant effect on production of at least one of the crops. The greatest significance in the N3SLS results is for the three crops (tomatoes, bell peppers, and melons) grown by most farmers. One explanation for why human capital does not explain significant differences in onion and eggplant production is that only the best farmers successfully produce those crops.

Significant (neutral) technological change was suggested for the two most important crops also. That is, since the constant term corresponds to 1980, the shift terms suggest a significant positive trend with annual weather variations for both tomatoes and bell peppers.

In addition to showing that crop-specific production elasticities can be estimated with aggregate input data, this example also demonstrates other useful applications of the model and approach. One is to estimate unobserved input allocations to crops. That is, using estimated coefficients and first-order conditions, the unobserved allocation is estimated by

$$\hat{x}_{ijkt} = \hat{\alpha}_{jk} \frac{r_{ikt}}{w_{jt}}, \quad j = J_{f+1}, \ldots, J_v$$

where $\hat{\alpha}_{jk}$ denotes the estimate of α_{jk}. Some further uses of the model based on these calculations are to determine input misallocations by individual farmers and to develop norms or recommendations for input use. For example, for each unconstrained input, the magnitude of

$$x_{ijt} - \sum_{k=1}^{K} \hat{x}_{ijkt} \tag{30}$$

is a measure of whether a farmer overuses or underuses the input for any particular observation. Alternatively, shift variables can be used in equation (23) to see if such misallocation persists over time. For constrained inputs, the difference in estimated marginal value of products is

$$\hat{\alpha}_{jk} \frac{r_{ikt}}{x_{ijkt}} - \hat{\alpha}_{j\kappa} \frac{r_{i\kappa t}}{x_{ij\kappa t}}. \tag{31}$$

If this expression is positive, then the constrained input is overallocated to crop κ versus crop k (and vice versa). Or, (31) can be rewritten in the form of (24) as

[9] The arguments presented here were based on numerous conversations with farm advisors in the Arava, farmers on the Moshav itself, and Jewish Agency officials responsible for Arava settlement.

Table 1. Estimated Model

Equation/coefficient	Two-Stage Least Squares Estimate	Two-Stage Least Squares t-Ratio	Three-Stage Least Squares Estimate	Three-Stage Least Squares t-Ratio
Tomato production function				
Constant term	7.545	10.56	8.623	13.27
Water elasticity	0.0374	2.78	0.0214	1.91
Other purchased input elasticity	0.0865	1.49	0.0173	0.34
Management	0.0585	1.73	0.0377	1.31
1977 shift	−0.2234	−2.52	−0.2248	−2.95
1978 shift	−0.0398	−0.46	−0.0094	−0.13
1979 shift	−0.2066	−2.47	−0.2732	−3.82
Bell pepper production function				
Constant term	4.679	12.94	4.534	13.64
Water elasticity	0.0457	7.14	0.0509	9.32
Other purchased input elasticity	0.2905	9.12	0.3087	11.01
Management	0.0583	2.86	0.0246	1.45
1977 shift	−0.5084	−10.39	−0.5505	−13.23
1978 shift	−0.5470	−11.44	−0.5175	−12.61
1979 shift	−0.2839	−5.64	−0.2909	−6.84
Onion production function				
Constant term	2.888	2.79	2.496	2.49
Water elasticity	0.0509	3.49	0.0551	4.28
Other purchased input elasticity	0.3990	4.34	0.4309	5.04
Management	0.0409	1.01	0.0200	0.61
1977 shift	0.3757	4.04	0.4975	7.23
1978 shift	0.3426	2.23	0.3768	3.13
1979 shift	0.3722	3.32	0.4463	5.14
Melon production function				
Constant term	6.368	22.40	6.196	24.87
Water elasticity	0.0049	1.29	0.0085	2.64
Other purchased input elasticity	0.0714	3.52	0.0807	4.39
Management	0.0507	1.03	0.0624	1.73
1977 shift	−0.0844	−0.64	0.0329	0.34
1978 shift	0.2826	2.66	0.2060	2.67
1979 shift	−0.0391	−0.33	−0.0083	−0.10
Eggplant production function				
Constant term	4.817	7.52	4.840	7.83
Water elasticity	0.0788	8.64	0.0776	9.57
Other purchased input elasticity	0.2225	4.18	0.2233	4.51
Management	0.0849	2.21	0.0310	0.96
1977 shift	−0.0858	−0.75	−0.0172	−0.18
1978 shift	0.1203	1.18	0.3128	3.66
1979 shift	−0.2396	−2.04	0.0305	0.33
Constant term of equation (24)				
k, bell peppers; κ, tomatoes	2.0264	16.39	2.1984	21.29
k, onions; κ, tomatoes	2.0776	9.11	2.3160	10.74
k, melons; κ, tomatoes	1.9227	15.57	2.0937	19.97
k, eggplants; κ, tomatoes	1.0054	7.68	1.1093	8.92

$$(32)\quad \ln x_{ijkt} - \ln \hat{\alpha}_{jk} + \ln \hat{\alpha}_{j\kappa} - \ln\left(\frac{r_{ikt}\, x_{ij\kappa t}}{r_{i\kappa t}}\right) - \left(\frac{\sigma_k{}^r - \sigma_\kappa{}^r}{2}\right),$$

where the latter term is the estimated constant term of equation (24) reported in table 1 for the random production/price case. By comparing the deviations in (30) and (31) with the standard error of the disturbance estimated for the respective equations (23) and (24), one can see whether each farmer is within a confidence interval around estimated profit-maximization relationships.

Conclusions

This paper introduces an approach for comprehensive consideration of multi-output production problems with the common technological characteristics and data availability faced in agriculture. It uses all available information from both technological and behavior assumptions in producing estimates of multi-output production functions where allocations of variable inputs among crops are unobserved. The empirical results show that the approach is practical and inexpensive and provides reasonable estimates.

The results also show that the popular single-equation, multiple-output production function approach is relatively unattractive by comparison. First, single-equation, multiple-output production functions impose just as many restrictions as this approach (even though the restrictions are not recognized explicitly). Second, the related information is not exploited for estimation. Finally, the results are not practical since the effect of input allocations cannot be estimated. Hence, the results cannot be used in analyzing input-allocation decisions or making related recommendations.

The approach presented here has wide potential for examining agricultural production problems. For example, by varying the technological specification, one can examine nonneutrality in outputs as well as inputs. The estimates in table 1 suggest that the production-possibilities curve for, say, bell peppers and melons has been shifting in favor of bell peppers. In addition, by extending the human capital specification, one can examine the relative effects of farmers' worker ability versus allocative ability. One approach would be to regress squared estimated disturbances in first-order conditions on human capital variables. Similarly, the approach developed for adapting specification to data observability offers rich possibilities for empirical investigation of other unexplored issues.

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References

Green, H. A. J. *Aggregation in Economic Analysis: An Introductory Survey*. Princeton NJ: Princeton University Press, 1964.

Hall, R. E. "The Specification of Technology with Several Kinds of Output." *J. Polit. Econ.* 81(1973):878–92.

Hausman, Jerry A. "An Instrumental Variable Approach to Full Information Estimators for Linear and Certain Nonlinear Econometric Models." *Econometrica* 43 (1975):727–38.

Just, Richard E. "Risk Aversion under Profit Maximization." *Amer. J. Agr. Econ.* 57(1975):347–52.

Just, Richard E., and Rulon D. Pope. "On the Relationship of Input Decisions and Risk." *Risk and Uncertainty in Agricultural Development*, ed. James A. Roumasset, Jean-Marc Boussard, and Inderjit Singh. College Laguna, Philippines: SEARCA Publications, 1979.

Klein, L. R. "The Use of Cross-Section Data in Econometrics with Application to a Study of Production of Railroad Services in the United States." Mimeographed. Washington DC: National Bureau of Economic Research, 1947.

Lau, L. J. "Profit Functions of Technologies with Multiple Inputs and Outputs." *Rev. Econ. and Statist.* 54(1972):281–89.

Marschak, J., and William H. Andrews, Jr. "Random Simultaneous Equations and the Theory of Production." *Econometrica* 12(1944):143–206.

Mittlehammer, Ron C., Scott C. Matulich, and D. Bushaw. "An Implicit Form of Multiproduct-Multifactor Production Functions." *Amer. J. Agr. Econ.* 63(1981):164–68.

Mundlak, Y. "Transcendental Multiproduct Production Functions." *Int. Econ. Rev.* 5(1964):273–84.

Mundlak, Yair, and Irving Hoch. "Consequences of Alternative Specifications in Estimation of Cobb-Douglas Production Functions." *Econometrica* 33 (1965):814–28.

Pfouts, Ralph W. "The Theory of Cost and Production in the Multiproduct Firm." *Econometrica* 39(1961): 65–68.

Powell, A. A., and F. H. Gruen. "The Constant Elasticity of Transformation: Frontier and Linear Supply System." *Int. Econ. Rev.* 9(1968):315–28.

Shumway, C. Richard, and Anne A. Chang. "Texas Field Crop Supply Response: Application of a Linear CET Supply Model." Paper presented at the AAEA annual meeting, Blacksburg VA, 8 Aug. 1978.

Shumway, C. Richard, Rulon D. Pope, and Elizabeth K. Nash. "Allocatable Fixed Inputs and Jointness in Agricultural Production: Implications for Economic Modeling." *Amer. J. Agr. Econ.*, in press.

Weaver, Robert D. "Multiple Input, Multiple Output Production Choices and Technology in the U.S. Wheat Region." *Amer. J. Agr. Econ.* 65(1983):45–56.

Whittaker, James K. "The Use of Linear Restrictions in Estimation of Crop Acreage Response: Homogeneity and the Constant Elasticity of Transformation." Paper presented at the AAEA annual meeting, Blacksburg VA, 8 Aug. 1978.

Zellner, A., J. Kmenta, and J. Dreze. "Specification and Estimation of Cobb-Douglas Production Function Models." *Econometrica* 34(1966):784–95.